A NOTE ON ERLANG'S FORMULAS

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ABSTRACT

A service system consisting of $n = \infty$ lines, is considered. Formulas (2) giving the probabilities of various states of the system were obtained by the assumption that the incoming stream is stationary, orderly, and without after-effects. It is proved in this note that these formulas hold as well when the incoming stream is not orderly.

1. **Introduction.** We confine ourselves to the terminology use in Khintchine's monograph [1].

Let us consider a service system consisting of n lines into which enters a random stream of calls. The durations of the call service are assumed to be identically distributed random variables which are independent both of each other and of the current of the incoming stream.

Denote by π_k (t) the probability that exactly k lines are occupied by the service at the moment t. If the incoming stream is simple (a Poisson stream) then the limits

$$\pi_k = \lim_{t\to\infty} \pi_k(t), \qquad (k = 0, 1, \cdots, n)$$

exist and are independent of the initial probabilities $\pi_k(0)$ $(0 \leqq k \leqq n)$. These limits are given by the formulas

$$(1) \qquad \pi_k = \frac{(\mu s)^k}{k!} \Big/ \sum_{i=0}^{n} \frac{(\mu s)}{i!} \qquad (k = 0, 1, \cdots, n)$$

where μ is the intensity of the stream and s is the mean duration of the service of one call.

Formulas (1) were first obtained by Erlang [2] under the assumption that the duration of the service of a call is exponentially distributed; however, it was shown after wards [3] - [5] that they remain true for an arbitrary distribution of the service duration.

Khintchine has also studied Erlang's problem for $n = \infty$. He proved that in this case the following meaning must be attributed to Erlang's formulas (1):

$$(2) \qquad \pi_k = e^{-\mu s} \frac{(\mu s)^k}{k!} \qquad (k = 0, 1, \cdots).$$

The aim of our note is to show that the last formulas are correct also for the case when the incoming stream is an arbitrary stationary stream without after-effects (i.e. not necessarily orderly) with a finite intensity μ and the distribution $H(x)$ of the duration ξ of service of one call

$$(3) \qquad H(x) = P(\xi > x), \qquad H(0) = 1$$

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satisfy the only natural restriction

$$(4) \qquad 0 < s = -\int_0^\infty x dH(x) = \int_0^\infty H(x)dx < \infty .$$

2. **An auxiliary proposition.** We will need the following elementary proposition from the theory of limits:

LEMMA. *Let the function $f(x)$ be differentiable in the closed interval $[a, 1]$ and infinitely differentiable in the half closed interval $[a, 1)$, and let in addition*

$$(5) \qquad f^{(k)}(x) \geqq 0 \qquad (k \geqq 2;\ a \leqq x < 1).$$

Let $\phi(t)$ $(t > 0)$ be a function such that

$$(6) \qquad \lim_{t \to \infty} \phi(t) = s > 0.$$

Then as $t \to \infty$

$$(7) \qquad t\left\{f(1) - f\left[1 - \frac{\phi(t)}{t}\right]\right\} \to sf'(1)$$

and for every positive integer n

$$(8) \qquad f^{(n+1)}\left[1 - \frac{\phi(t)}{t}\right] \Big/ t^n \to 0 .$$

Proof. From (6) follows that for large enough t

$$a < 1 - \frac{\phi(t)}{t} < 1.$$

Using the Mean Value Theorem we obtain

$$t\left\{f(1) - f\left[1 - \frac{\phi(t)}{t}\right]\right\} = \phi(t)f'\left[1 - \theta\frac{\phi(t)}{t}\right],$$

where $0 < \theta < 1$. The expression (7) follows now, since by assumption (5) the derivative $f'(x)$ is continuous from the left at $x = 1$.

Let us prove (8) by induction on n. $f''(x)$ is not decreasing by (5), hence

$$f'(1) - f'\left[1 - \frac{\phi(t)}{t}\right] \geqq \phi(t)f''\left[1 - \frac{\phi(t)}{t}\right] \Big/ t \geqq 0;$$

these inequalities prove (8) for $n = 1$.

Again, from (6) follows that for large enough t

$$\phi(2t) < 2\phi(t)$$

and therefore

$$1 - \frac{\phi(t)}{t} < 1 - \frac{\phi(2t)}{2t} .$$

On the other hand, $f^{(k+1)}(x)$ is non decreasing by (5). Thus another application of the Mean Value Theorem will yield

$$f^{(k)}\left[1 - \frac{\phi(2t)}{2t}\right] - f^{(k)}\left[1 - \frac{\phi(t)}{t}\right] \geqq \frac{2\phi(t) - \phi(2t)}{2} \cdot \frac{f^{(k+1)}\left[1 - \frac{\phi(t)}{t}\right]}{t} \geqq 0,$$

hence a fortiori

$$\frac{f^{(k)}\left[1-\frac{\phi(2t)}{2t}\right]}{(2t)^{k-1}} \geqq \frac{2\phi(t)-\phi(2t)}{2^k} \cdot \frac{f^{(k+1)}\left[1-\frac{\phi(t)}{t}\right]}{t^k} \geqq 0 .$$

Thus we proved that if (8) holds for $n = k-1$ then it holds also for $n = k$.

3. **Proof of formulas** (2). We prove the validity of the formulas (2) under assumption that the stream of calls began at the moment $t_0 = 0$, i.e., $\pi_0(0) = 1$ (The proof would be only a little more difficult for arbitrary preliminary data).

Let $v(t)$ denote the probability that the service of a call that occurs in the time interval $[0,t)$ will be determined in the same interval. It will be shown that

$$v(t) = 1 - \frac{1}{t}\int_0^t H(x)dx \tag{9}$$

and therefore $v(t)$ is independent of the number of calls which occur in the interval $[0,t)$.

For the proof, let us assume that in the interval $[0,t)$ exactly m calls occured and let η be the moment that one of the calls, taken at random, appeared. Clearly

$$v(t) = P(\eta + \xi < t), \tag{10}$$

where ξ is the duration of service of a call.

In [6] it was shown that if the stream is stationary and without after-effects then, whatever is m, the random variable η is uniformly distributed in the interval $[0,t)$. Since by our assumptions the variables η and ξ are independent, equation (9) is easily deduced from (3) and (10).

Denote by $P_m(t)$ the probability that exactly m calls will occur in the interval $[0,t)$. Then, by the above remarks,

$$\pi_k(t) = \sum_{m=0}^{\infty} P_{m+k}(t)\binom{m+k}{k}[1-v(t)]^k v^m(t)$$

or

$$\pi_k(t) = \frac{[1-v(t)]^k}{k!}\sum_{m=0}^{\infty}\frac{v^m(t)}{m!}(m+k)!\,P_{m+k}(t).$$

Consider the generating function of the stream

$$F(t,x) = \sum_{m=0}^{\infty} P_m(t)x^m, \qquad (|x| \leq 1).$$

Since

$$(m+k)!\,P_{m+k}(t) = \frac{\partial^{m+k}F(t,0)}{\partial x^{m+k}} = \frac{\partial^m}{\partial x^m}\left(\frac{\partial^k F(t,x)}{\partial x^k}\right)\Big|_{x=0},$$

it is easy to see that the probabilities $\pi_k(t)$ can also be expressed by

$$\pi_k(t) = \frac{[1-v(t)]^k}{k!} \frac{\partial^k F[t, v(t)]}{\partial x^k}.$$

Finally, if we put

$$\phi(t) = \int_0^t H(x)dx, \tag{11}$$

we conclude from (9) that

$$\pi_k(t) = \frac{\phi^k(t)}{k!} \frac{1}{t^k} \frac{\partial^k F[t, 1-\phi(t)/t]}{\partial x^k}. \tag{12}$$

It is known, [1], that the generating function of a stationary stream without after-effects has the form

$$F(t,x) = \exp\{\lambda t[f(x) - 1]\}, \tag{13}$$

where $\lambda > 0$ and $p_i \geqq 0$, ($\sum_{i=1}^{\infty} p_i = 1$) are constants and

$$f(x) = \sum_{i=1}^{\infty} p_i x^i, \qquad (|x| \leqq 1). \tag{14}$$

Moreover, the intensity of the stream is given by

$$\mu = \sum_{i=1}^{\infty} i p_i = \lambda f'(1). \tag{15}$$

Assuming that the intensity is finite we get

$$1 \leqq f'(1) < \infty. \tag{16}$$

It is easy to see that the partial derivatives of the generating function (13) are given by

$$\frac{\partial^k F(t,x)}{\partial x^k} = F(t,x)\{[\lambda t f'(x)]^k + \sum_{m=1}^{k-1} (\lambda t)^m [\sum A(i_1, \cdots, i_m) \prod_{s=1}^{m} f^{(i_s)}(x)]\}, \tag{17}$$

where the inner summation is over various systems of positive integers $(i_1, \cdots, i_m)$ that satisfy

$$i_1 + \cdots + i_m = k, \tag{18}$$

and $A(i_1, \cdots, i_m)$ are constants that depend only on the indices $(i_1, \cdots, i_m)$. (Note that it ts possible to give the more detailed formula

$$\frac{\partial^k F(t,x)}{\partial x^k} = k!F(t,x) \sum_{m=1}^{k} \frac{(\lambda t)^m}{m!} \{\sum \prod_{s=1}^{m} \frac{f^{(i_s)}(x)}{i_s!}\},$$

where the summation is over all the ordered systems of positive integers satisfying (18); for our purpose formula (17) is sufficient).

Taking into account that in view of (18)

$$\prod_{s=1}^{m} t^{i_s - 1} = t^{k-m},$$

we conclude by (12) and (17) that

$$(19)\quad \pi_k(t) = F(t,x)\,\frac{\phi^k(t)}{k\,!}\left\{[\lambda f'(x)]^k + \sum_{m=1}^{k-1}\lambda^m\left[\Sigma A(i_1,\cdots,i_m)\prod_{s=1}^{m}\frac{f^{(i_s)}(x)}{t^{i_s-1}}\right]\right\},$$

where we put for brevity

$$x = 1 - \frac{\phi(t)}{t}.$$

Our functions $f(x)$ and $\phi(t)$ satisfy all the conditions of the Lemma: $f(x)$ by (14) and (16), and $\phi(t)$ by (4) and (11). Since $f(1) = 1$, from (6), (7) and (13) follows that

$$\lim_{t\to\infty} F\left[t, 1 - \frac{\phi(t)}{t}\right] = e^{-s\lambda f'(1)} = e^{-\mu s}.$$

On the other hand, if $1 \leqq m \leqq k-1$, then every system of positive integers $(i_1,\cdots,i_m)$ that satisfies equation (18) contains at least one integer $i \geqq 2$. Therefore, because of (8),

$$\lim_{t\to\infty}\prod_{s=1}^{m}\frac{f^{(i_s)}\left[1 - \frac{\phi(t)}{t}\right]}{t^{i_s-1}} = 0$$

for all the systems $(i_1,\cdots,i_m)$ that paticipate in (19). Thus the second term in the braces of the expression (19), being a finite combination of terms that tend to zero, also tends to zero. This proves formulas (2), since

$$\lim_{t\to\infty}\phi^k(t) = s^k$$

and

$$\lim_{t\to\infty}\left[\lambda f'\left(1 - \frac{\phi(t)}{t}\right)\right]^k = [\lambda f'(1)]^k = \mu^k.$$

REMARK. Formulas (2) can be applied to a slightly more general situation. Let a finite or countable number of stationary streams without after-effects $x_i(t)$ with intensity μ_i $(i = 1,2,\cdots)$ enter a service system consisting of $n = \infty$ lines. Assume that the distribution of the duration of the call service is the same for all calls in the same stream but this distribution depends on the index i and it may vary from one stream to another.

Denote by s_i the mean duration of the service of a call which belongs to the stream $x_i(t)$, and assume that $s_i < \infty$ for every i, and

$$\sum_{i=1}^{\infty}\mu_i < \infty \quad , \quad \sum_{i=1}^{\infty}\mu_i s_i < \infty.$$

Since the total stream $x(t) = \sum_{i=1}^{\infty} x_i(t)$ is also finite, stationary and without after-effects, then formulas (2) hold also for it. It is clear that μ and s are determined by

$$\mu = \sum_{i=1}^{\infty} \mu_i , \qquad s = \frac{1}{\mu} \sum_{i=1}^{\infty} \mu_i s_i .$$

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